Indian Statistical Institute

Final Examination 2020-2021

Analysis III, B.Math Second Year

Date : 14.12.2020 Maximum Marks : 100 Duration : 3 Hours

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(1) (10 marks) Compute $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2+y^2)^2 \, dx \, dy$ by making a change of variables to polar coordinates.

(2) (15 marks) Compute (i) $\int_{-1}^{1} \int_{|y|}^{1} (x+y) \, dx \, dy$; and (ii) $\int_V x$, where V is the solid tetrahedron bounded by the coordinate planes and the first octant part of the plane $2x+3y+z=6$.

(3) (15 marks) Let $f \geq 0$ be a continuous function on a box $B \subseteq \mathbb{R}^n$ and let $a \in B$. Prove that if $f(a) > 0$, then

$$\int_B f > 0.$$

(4) (15 marks) Let Ω be an open subset of $\mathbb{R}^2$ that contains

$$R = \{(x,y) \in \mathbb{R}^2 : 0 \leq x+y \leq 1\}.$$

Suppose $f \in C^2(\Omega)$. Prove that there exists $t \in [0,1]$ such that

$$\int_R \frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial f}{\partial x}(t, 1-t) + f(0,0) - f(1,0).$$

[*Hint:* If $t \in [0,1]$, then $(t, 1-t)$ lies on the line segment joining points $(1,0)$ and $(0,1)$.]

(5) (15 marks) Let S be an oriented sphere and let F be a C^1-vector field on an open set containing S. Prove that

$$\int_S \mathrm{curl} F \cdot dS = 0.$$

(6) (15 marks) Prove that if $f \in C[0,1]$ and

$$\int_0^1 x^{2n} f(x) \, dx = 0,$$

for all $n \geq 1$, then $f \equiv 0$.

(7) (15 marks) Let $f \in C[0,1]$ and

$$f_n(x) := x^n f(x) \qquad (x \in [0,1]),$$

for all $n \geq 0$. Prove that $\{f_n\}_{n \geq 0}$ is uniformly convergent on $[0,1]$ if and only if $f(1) = 0$.